DISCONTINUITY OF THE LEMPERT FUNCTION OF THE SPECTRAL BALL

P. J. THOMAS, N. V. TRAO

ABSTRACT. We give some further criteria for continuity or discontinuity of the Lempert funtion of the spectral ball Ω_n, with respect to one or both of its arguments, in terms of cyclicity the matrices involved.

1. INTRODUCTION AND STATEMENT OF RESULTS

The spectral ball is the set of all $n \times n$ complex matrices with eigenvalues strictly smaller than one in modulus. It can be seen as the union of all the unit balls of the space of matrices endowed with all the possible operator norms arising from a choice of norm on the space $\mathbb{C}^n$. It contains many entire curves. As analogues of the Montel theorem cannot hold, several invariant objects in complex analysis exhibit discontinuity phenomena in this setting, first pointed out in [1].

The goal of this note is to give a few facts about discontinuities of the Lempert function (corresponding to the two-point Pick-Nevanlinna problem), to be compared with [11].

We fix some notation. Let $\mathcal{M}_n$ be the set of all $n \times n$ complex matrices. For $A \in \mathcal{M}_n$ denote by $sp(A)$ and $r(A) = \max_{\lambda \in sp(A)} |\lambda|$ the spectrum and the spectral radius of A, respectively. The *spectral ball* Ω_n is the set
$$\Omega_n = \{A \in \mathcal{M}_n : r(A) < 1\}.$$
The characteristic polynomial of the matrix A is
$$P_A(t) := \det(tI - A) =: t^n + \sum_{j=1}^{n} (-1)^j \sigma_j(A) t^{n-j},$$

where $I \in \mathcal{M}_n$ is the unit matrix. We define a map σ from $\mathcal{M}_n$ to $\mathbb{C}^n$ by $\sigma := (\sigma_1, \ldots, \sigma_n)$. The *symmetrized polydisk* is $\mathbb{G}_n := \sigma(\Omega_n)$ is a bounded domain in $\mathbb{C}^n$, which is hyperconvex [6] and, therefore, a complete hyperbolic domain, and taut. As is noted in the same paper

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or in [11, Proposition 7], $\sigma(A) = \sigma(B)$ if and only if there is an entire curve contained in Ω_n going through A and B.

For general facts about invariant (pseudo)distances and (pseudo)metrics, see for instance [8]. The Lempert function of a domain $D \subset \mathbb{C}^m$ is defined, for $z, w \in D$, as

$$l_D(z, w) := \inf\{|\alpha| : \alpha \in \mathbb{D} \text{ and } \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(0) = z, \varphi(\alpha) = w\}.$$

The Lempert function is upper semicontinuous and decreases under holomorphic maps, so for $A, B \in \Omega_n$,

$$(1.1) \qquad\qquad l_{\Omega_n}(A, B) \geq l_{\mathbb{G}_n}(\sigma(A), \sigma(B)).$$

On $\mathbb{G}_n$, the Lempert function is continuous, and the remark above about entire curves shows that $l_{\mathbb{G}_n}(\sigma(A), \sigma(B)) = 0$ if and only if $l_{\Omega_n}(A, B) = 0$.

For a Zariski dense open set of matrices, equality holds in (1.1) and the Lempert function is continuous. Recall that a matrix A is *cyclic* (or *non-derogatory*) if it admits a cyclic vector (see for instance [7]). As in [11], we denote by $C_{\sigma(A)}$ the companion matrix of the characteristic polynomial of A; A is cyclic if and only if it is conjugate to $C_{\sigma(A)}$.

Agler and Young [1] proved that if A and B are cyclic, any holomorphic mapping $\varphi \in \mathcal{O}(\mathbb{D}, \mathbb{G}_n)$ through $\sigma(A)$ and $\sigma(B)$ lifts to $\Phi \in \mathcal{O}(\mathbb{D}, \Omega_n)$ through A and B, so that in particular:

Proposition 1.1 (Agler-Young)**.** *If $A, B \in \Omega_n$ are cyclic, then*

$$(1.2) \qquad\qquad l_{\Omega_n}(A, B) = l_{\mathbb{G}_n}(\sigma(A), \sigma(B)).$$

Continuity of the Lempert function near such a pair (A, B) follows from the fact that cyclicity is an open condition, or from the following.

Proposition 1.2. *Let $A, B \in \Omega_n$.*

   (1) *The Lempert function l_{Ω_n} is continuous at (A, B) if and only if (1.2) holds.*
   (2) *If B is cyclic, and the function $l_{\Omega_n}(., B)$ is continuous at A, then (1.2) holds.*

The proofs are given in Section 2.

We see that when B is cyclic, continuity of the Lempert function with respect to both variables reduces to continuity with respect to the first variable. We now study this partial continuity for iself.

Theorem 1.3. *Let $A \in \Omega_n$. If A is not cyclic (or* derogatory*), then there exists a matrix $B \in \Omega_n$ such that the function $l_{\Omega_n}(., B)$ is not continuous at A.*

The proof is given in Section 3.

Conversely, suppose that A is cyclic. If B is cyclic, Proposition 1.1 settles the question. If not, the converse holds in several interesting cases.

Proposition 1.4. *Let $A, B \in \Omega_n$, A cyclic. If B has only one eigenvalue, or $n \leq 3$, then the function $l_{\Omega_n}(., B)$ is continuous at A.*

This is proved in Section 4.

We conjecture that the additional hypotheses on B or n can be dispensed with.

When A is derogatory, one may wonder which matrices B make the function $l_{\Omega_n}(., B)$ continuous. The case where $A = tI$, treated in [11], suggests that discontinuity is generic. We give some relevant examples in Section 5. In Section 6, we apply our results to compare the Lempert and Green functions on the spectral ball.

2. PROOF OF PROPOSITION 1.2.

2.1. **Proof of (i), direct part.** Since the cyclic matrices are dense in Ω_n then there exist $A_j, B_j \in \mathcal{C}_n$ such that $A_j \to A, B_j \to B$. By continuity of l_{Ω_n} at (A, B) we get that $l_{\Omega_n}(A_j, B_j) \xrightarrow{j \to \infty} l_{\Omega_n}(A, B)$.

On the other hand $l_{\Omega_n}(A_j, B_j) = l_{\mathbb{G}_n}(\sigma(A_j), \sigma(B_j))$. By tautness of the domain $\mathbb{G}_n$ we have $l_{\mathbb{G}_n}(\sigma(A_j), \sigma(B_j)) \xrightarrow{j \to \infty} l_{\mathbb{G}_n}(\sigma(A), \sigma(B))$. This implies that $l_{\Omega_n}(A, B) = l_{\mathbb{G}_n}(\sigma(A), \sigma(B))$.

2.2. **Proof of (i), converse part.** Assume $l_{\Omega_n}(A, B) = l_{\mathbb{G}_n}(\sigma(A), \sigma(B))$.

Let $(A_j, B_j) \subset \Omega_n$ be such that $(A_j, B_j) \xrightarrow{j \to \infty} (A, B)$ and

$$\lim_{j \to \infty} l_{\Omega_n}(A_j, B_j) = a := \liminf_{(X,Y) \to (A,B)} l_{\Omega_n}(X, Y).$$

We have

$$l_{\Omega_n}(A_j, B_j) \geq l_{\mathbb{G}_n}(\sigma(A_j), \sigma(B_j)) \to l_{\mathbb{G}_n}(\sigma(A), \sigma(B)),$$

and hence $a \geq l_{\mathbb{G}_n}(\sigma(A), \sigma(B)) = l_{\Omega_n}(A, B)$. Then l_{Ω_n} is lower semicontinuous at (A, B). Since l_{Ω_n} is always upper semicontinuous, it is continuous at (A, B).

2.3. **Proof of (ii).** We only need to repeat the proof of the direct part of (i), taking $B_j = B$ for all j. Then we only use the continuity of l_{Ω_n} in the first variable. □

3. PROOF OF THEOREM 1.3.

We shall need a theorem by Bharali [3].

Theorem 3.1 (Bharali). *Let $F \in \mathcal{O}(\mathbb{D}, \Omega_n), n \geq 2$, and let $\zeta_1, \zeta_2 \in \mathbb{D}$. Write $W_j = F(\zeta_j), j = 1, 2$. If $\lambda \in sp(W_j)$, then let $m(\lambda)$ denote the multiplicity of λ as a zero of the minimalpolynomial of W_j. Then*

$$(3.1) \quad \max \left\{ \max_{\mu \in sp(W_2)} \prod_{\lambda \in sp(W_1)} \left| \frac{\mu - \lambda}{1 - \bar{\lambda}\mu} \right|^{m(\lambda)} ; \max_{\lambda \in sp(W_1)} \prod_{\mu \in sp(W_2)} \left| \frac{\lambda - \mu}{1 - \bar{\mu}\lambda} \right|^{m(\mu)} \right\}$$

$$\leq \left| \frac{\zeta_1 - \zeta_2}{1 - \overline{\zeta_2}\zeta_1} \right|.$$

Now, let A be derogatory. The idea will be to construct a matrix B a short distance away from A, in a direction which belongs to the kernel of the differential map of σ at A, but where the Kobayashi Royden pseudometric doesn't vanish. Compare with the proof of Proposition 3 and in particular Lemma 8 in [10].

Since A is derogatory, at least two of the eigenvalues of A are equal, say to λ. Applying the automorphism of Ω_n given by $M \mapsto (\lambda I - M)(I - \bar{\lambda}M)^{-1}$, we may assume that $\lambda = 0$. Since the map $A \to P^{-1}AP$ is a linear automorphism of Ω_n for any $P \in \mathcal{M}_n^{-1}$, we may also assume that A is in Jordan form. In particular,

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix},$$

where $A_0 \in \mathcal{M}_m$, $2 \leq m \leq n$, $sp(A_0) = \{0\}$, $A_1 \in \mathcal{M}_{n-m}$, $0 \notin sp(A_1)$. Furthermore, there is a set $J \subsetneq \{2, \ldots, m\}$, possibly empty, such that $a_{j-1,j} = 1$ for $j \in J$, and all other coefficients $a_{ij} = 0$ for $1 \leq i, j \leq m$. Denote $0 \leq r := \#J = \text{rank}A_0 \leq m - 2$ and k is the multiplicity of 0 as a zero of the minimal polynomial of $A_0, 0 < k \leq r + 1 < m$.

We set

$$X := \begin{pmatrix} X_0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_n,$$

where $X_0 = (x_{ij})_{1 \leq i,j \leq m}$ is such that $x_{j-1,j} = -1$ for $j \in \{2, \ldots, m\} \setminus J$, $x_{m1} = 1$, and $x_{ij} = 0$ otherwise.

For $\delta > 0$ is small enough we set

$$B = A + \delta X = \begin{pmatrix} B_0 & 0 \\ 0 & A_1 \end{pmatrix},$$

where $B_0 = A_0 + \delta X_0$.

We begin by computing $\sigma_j(B_0)$, $1 \le j \le m$. Expanding with respect to the first column, we see that

$$\det(tI - B_0) = t^m + (-1)^{m-1}\delta^{m-r}. \tag{3.2}$$

The m distinct roots $\lambda_1, \lambda_2, \cdots, \lambda_m$ of this polynomial are the eigenvalues of B_0, and the multiplicity of λ_j as a zero of the minimal polynomial of B_0 is $1, 1 \le j \le m$. Comparing the respective coefficients of both sides, it follows that

$$\sigma_j(B_0) = \begin{cases} 0, & 1 \le j \le m-1 \\ \delta^{m-r}, & j = m \end{cases}. \tag{3.3}$$

Let $sp(A_1) = \{\mu_1, \mu_2, \cdots, \mu_s\}$, with the multiplicity of μ_j as a zero of the minimal polynomial of A_1 denoted by $m_j, 1 \le j \le s$.

Consider now $\varphi \in \mathcal{O}(\mathbb{D}, \Omega_n)$ and $\zeta \in \mathbb{D}$ such that $\varphi(0) = A, \varphi(\zeta) = B$. Then, by applying (3.1), we obtain that

$$|\zeta| \ge \max\{|\lambda_1 \cdots \lambda_m \mu_1^{m_1} \cdots \mu_s^{m_s}|; |\lambda_j|^k \prod_{i=1}^{s} l_{\mathbb{D}}(\lambda_j, \mu_i)^{m_i}, 1 \le j \le m\}.$$

Using this and (3.2) with δ small enough we have

$$l_{\Omega_n}(A, B) \ge C \cdot \delta^{\frac{m-r}{m}k}, \text{ where } C \text{ is a constant.} \tag{3.4}$$

Take a sequence of cyclic matrices $A_0^j \to A_0$. If we consider the matrices

$$A^j := \begin{pmatrix} A_0^j & 0 \\ 0 & A_1 \end{pmatrix} \in \Omega_n,$$

then $A^j \to A$ as $j \to \infty$.

Define the map $f : \Omega_m \to \Omega_n$ by

$$f(M) = \begin{pmatrix} M & 0 \\ 0 & A_1 \end{pmatrix}.$$

Since $f(A_0^j) = A^j; f(B_0) = B$, we have

$$l_{\Omega_n}(A^j, B) \le l_{\Omega_m}(A_0^j, B_0). \tag{3.5}$$

Since A_0^j, B_0 are cyclic, we have

$$l_{\Omega_m}(A_0^j, B_0) = l_{\mathbb{G}_m}(\sigma(A_0^j), \sigma(B_0)) \tag{3.6}$$

Since $\mathbb{G}_m$ is a taut domain, $l_{\mathbb{G}_m}$ is a continuous function. Thus

$$l_{\mathbb{G}_m}(\sigma(A_0^j), \sigma(B_0)) \to l_{\mathbb{G}_m}(\sigma(A_0), \sigma(B_0)). \tag{3.7}$$

On the other hand, we can find $R > 0$ such that $\mathbb{B}(0, R) \subset \mathbb{G}_m$, where $\mathbb{B}(0, R)$ denotes the Euclidean ball with center at 0 and radius R. For

δ chosen small enough, $\sigma(B_0) \in \mathbb{B}(0, R)$. By the definition of Lempert function and [8, Proposition 3.1.10], we conclude that

$$(3.8) \quad l_{\mathbb{G}_m}(\sigma(A_0), \sigma(B_0)) \leq l_{\mathbb{B}(0,R)}(\sigma(A_0), \sigma(B_0))$$

$$= l_{\mathbb{B}(0,R)}((0, \cdots, 0), (0, \cdots, \delta^{m-r})) = \frac{\delta^{m-r}}{R}.$$

Combining (3.4),(3.5),(3.6),(3.7) and (3.8), we have

$$l_{\Omega_n}(A, B) > l_{\Omega_n}(A^j, B)$$

when δ is small enough and j is large enough. It implies the discontinuity of the Lempert function $l_{\Omega_n}(., B)$ at the point A. $\qquad\square$

Note that we have proved a slightly stronger statement than the proposition : for A to be cyclic, it is enough that the function $l_{\Omega_n}(., B)$ be continuous at A for all B in some neighborhood of A.

4. Proof of Proposition 1.4

4.1. **Lifting maps.** We need the following generalization of [1, Theorem 2.8].

Proposition 4.1. *Let $B \in \mathcal{M}_n(\mathbb{C})$ be a nilpotent matrix. Then there exists a linear map Θ_B from the set of analytic maps from $\mathbb{D}$ to $\mathbb{C}^n$, depending only on the values at 0 of the first $n-1$ derivatives of the coordinates of the map, such that :*

Given $\zeta_0 \in \mathbb{D}$, $A \in \Omega_n$ a cyclic matrix, and φ a holomorphic map from $\mathbb{D}$ to $\mathbb{G}_n$ such that $\varphi(0) = \sigma(B)$, $\varphi(\zeta_0) = \sigma(A)$, then there exists $\tilde{\varphi}$ a holomorphic map from $\mathbb{D}$ to Ω_n such that $\sigma \circ \tilde{\varphi} = \varphi$, $\tilde{\varphi}(0) = B$ and $\tilde{\varphi}(\zeta_0) = A$ if and only if $\Theta_B(\varphi) = 0$.

Notice that in the case where B is cyclic, σ is of maximal rank at B, so that $\Theta_B = 0$, and conversely it is known [1] that any map φ will admit a lifting through cyclic matrices.

Finally, the case where B admits a single (arbitrary) eigenvalue easily reduces to the nilpotent case by using the automorphism $\Phi_\lambda(M) = (\lambda I_n - M)(I_n - \bar{\lambda}M)^{-1}$, of the spectral ball.

Proof of Proposition 1.4 in the case of a single eigenvalue.

It is enough to prove that $\ell := \liminf_{M \to A} l_{\Omega_n}(M, B) \geq l_{\Omega_n}(A, B)$. Suppose that $A_j \to A$ ($j \geq 1$), and that $\tilde{\varphi}_j \in \mathcal{O}(\mathbb{D}, \Omega_n)$ are holomorphic maps and $\zeta_j \in \mathbb{D}$) such that $\tilde{\varphi}_j(0) = B$, $\tilde{\varphi}_j(\zeta_j) = A_j$, $\lim_{j \to \infty} |\zeta_j| = \ell$. Consider $\varphi_j := \sigma \circ \tilde{\varphi}_j$, and (by Montel's Theorem) extract a subsequence converging to $\varphi \in \mathcal{O}(\mathbb{D}, \mathbb{G}_n)$ and such that $\zeta_j \to \zeta_0$.

By the necessary condition in Proposition 4.1, $\Theta_B(\varphi_j) = 0$; $\varphi_j(0) = 0$ and $\varphi_j(\zeta_j) = \sigma(A_j)$. So compact convergence implies that $\varphi(0) = 0$,

$\varphi(\zeta_0) = \sigma(A)$ and $\Theta_B(\varphi) = 0$. By the sufficiency part of Proposition 4.1, there exists $\tilde\varphi \in \mathcal{O}(\mathbb{D}, \Omega_n)$ such that $\tilde\varphi(0) = B$, $\tilde\varphi(\zeta_0) = A$. Therefore $l_{\Omega_n}(A, 0) \le |\zeta_0| = \ell$. $\qquad\qquad\square$

4.2. **Proof of Proposition 4.1.** It will be enough to find a map $\tilde\varphi$ that satisfies the conclusion only with $\tilde\varphi(0) = B'$ and $\tilde\varphi(\zeta_0) = A'$, where B', A' are conjugate to B, A respectively, as in [1, Proof of Theorem 2.1]. Furthermore, any cyclic matrix with the same spectrum as A will be conjugate to A. So it is enough to check that $\tilde\varphi(0) = B$, where B is in Jordan form, and $\tilde\varphi(\zeta)$ is cyclic for $\zeta \ne 0$.

We use the following notations: $B = (b_{ij})_{1 \le i,j \le n}$. For two integers $k \le l$, $[k..l] := \{i \in \mathbb{Z} : k \le i \le l\}$. Let r stand for the rank of B. Write

$$F_0 := \{j : b_{ij} = 0 \text{ for } 1 \le i \le n\} := \{1 = b_1 < b_2 < \cdots < b_{n-r}\}.$$

For $j \in F_1 := [1..n] \setminus f_0$, $v_{j-1,j} = 1$, $v_{ij} = 0$ for $i \ne j-1$. We can choose the Jordan form so that $b_{l+1} - b_l$ is increasing for $1 \le l \le n - r$, with the convention $b_{n-r+1} := n + 1$.

Let us give the differential conditions satisfied by a map $\sigma \circ \tilde\varphi$ at 0. To do so, we must study the homogeneity of the functions $\sigma_i(B+M)$ in terms of the entries of M. Let $M = (m_{kl})_{1 \le k,l \le n}$. For $E \subset \{1, \ldots, n\}$, denote $M_E := (m_{kl})_{k,l \in E}$.

$$(4.1) \quad \sigma_i(B + M) = \sum_{E \subset \{1,\ldots,n\}, \#E=i} \det(B + M)_E$$

$$= \sum_{E \subset \{1,\ldots,n\}, \#E=i} \sum_{\theta \in \mathcal{S}(E)} (-1)^{\mathrm{sgn}(\theta)} \prod_{e \in E} (b_{e,\theta(e)} + m_{e,\theta(e)}),$$

where $\mathcal{S}(E)$ stands for the permutation group of E. This is a polynomial of degree $\le i$ in the entries of M.

Lemma 4.2. *Let $d_i := 1 + \#(F_0 \cap [(n-i+2)..n])$. The lowest order terms of $\sigma_i(B + M)$ are of degree d_i (in the entries of M).*

Proof. The terms in this polynomial in (4.1) are of the form

$$\pm \prod_{e \in E_1} v_{e,\theta(e)} \cdot \prod_{e \in E_2} a_{e,\theta(e)},$$

where $E = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$. Non zero terms of degree $\#E_2$ have sets E_1 such that

$$(4.2) \qquad\qquad v_{e,\theta(e)} = 1 \quad \forall e \in E_1,$$

so $\theta(e) = e + 1$ and $e + 1 \in F_1$.

Write $J_k := [(b_k + 1)..(b_{k+1} - 1)]$, so that $F_1 = \cup_k J_k$ (some of those may be empty), and $E_{1,k} := \{e \in E_1 : e + 1 \in J_k\}$. If $E_{1,k} \neq \emptyset$, then $1 + \max E_{1,k} = \theta(\max E_{1,k}) \in E$, and since it cannot be in any of the J_l, it must belong to E_2. Thus

$$d := \#E_2 \geq \# \{k : E_{1,k} \neq \emptyset\}.$$

We use the partition $E = E_2 \cup \bigcup_k E_{1,k}$:

$$\#E = i = d + \sum_{k:E_{1,k}\neq\emptyset} \#E_{1,k} \leq d + \sum_{k:E_{1,k}\neq\emptyset} \#J_k$$

$$\leq d + \sum_{k=n-r-d+1}^{k=n-r} (b_{k+1} - b_k - 1) = (n+1) - b_{n-r-d+1}.$$

Therefore $b_{n-r-d+1} \leq (n+1) - i < n - i + 2$, which means that $[(n-i+2)..n]$ contains no more elements of F_0 than those greater or equal to $b_{n-r-d+2}$, of which there are $d - 1$. So $\# (F_0 \cap [(n - i + 2)..n]) \leq d - 1$, i. e. $\#E_2 \geq d_i$.

To show that this bound is attained, choose

$$E_1 := \{e \in [(n - i + 1)..(n - 1)] : e + 1 \in F_1\}.$$

Then $\#E_1 = i - d_i$, and we can define the permutation θ by $\theta(e) = e+1$, $n - i + 1 \leq e \leq n$, and $\theta(n) = n - i + 1$, so that $E = [(n - i + 1)..n]$ and $\#E = i$, as required. $\qquad\square$

Corollary 4.3. *If* $\varphi = (\varphi_1, \ldots, \varphi_n) = \sigma \circ \tilde{\varphi}$, *with* $\tilde{\varphi} \in \mathcal{O}(\mathbb{D}, \Omega_n)$, $\tilde{\varphi}(0) = B$, *then* $\varphi_i^{(k)}(0) = 0$, $0 \leq k \leq d_i - 1$.

This gives additional (differential) conditions whenever $d_i \geq 2$.

Conversely, given a map $\varphi \in \mathcal{O}(\mathbb{D}, \mathbb{G}_n)$ satisfying the conclusion of Corollary 4.3, let

$$\psi(\zeta) := \begin{pmatrix} 0 & f_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & \ddots & f_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & f_n \\ \psi_n & \psi_{n-1} & \cdots & \psi_2 & \psi_1 \end{pmatrix}.$$

Lemma 4.4. *For a matrix ψ as above,* $\sigma_i(\psi) = (-1)^{i+1}\psi_i \prod_{k=n-i+2}^{n} f_k$.

Proof. Expanding with respect to the last row,

$$\det(XI_n - M) = X^n + \sum_{j=1}^{n}(-1)^{j+n}(-\psi_{n+1-j})X^{j-1} \prod_{k=j+1}^{n} (-f_k),$$

and we find the coeficient of $(-1)^i X^{n-i}$ by setting $j = n - i + 1$. $\qquad\square$

We define a lifting by taking ψ with $f_j(\zeta) = 1$ for $j \in F_1$, $f_j(\zeta) = \zeta$ for $j \in F_0$, $\psi_j(\zeta) = (-1)^{i+1}\zeta^{-d_j+1}\varphi_j(\zeta)$. Those coefficients are holomorphic, and $\psi(0) = B$, since φ satisfies the conclusion of Corollary 4.3. Lemma 4.4 shows that $\sigma \circ \psi = \varphi$. Finally, for any $\zeta \neq 0$, the matrix $\psi(\zeta)$ is cyclic, and $\sigma \circ \psi(\zeta_0) = \sigma(A)$, so we can modify ψ as indicated at the beginning of the section, to obtain a lifting through A.

4.3. **The cases** $n = 2$ **or** 3. The only cases to deal with are those were B is derogatory (non cyclic). As above, we can reduce ourselves to the case where one eigenvalue of B is 0.

When $n = 2$, the only derogatory matrices are scalar, therefore we only need to consider the case $B = 0$ and Proposition 4.1 (or indeed, [1, Theorem 2.8] settles the question. When $n = 3$ and B is derogatory, either it has a single eigenvalue, or it has two distinct eigenvalues, one of which has an eigenspace of dimension 2. We then may assume that $\dim \ker B = 2$, and, say,

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix},$$

for some $\lambda_1 \in \mathbb{D}$.

If $\varphi = \sigma \circ \tilde{\varphi}$, simple determinant calculations show that

(4.3) $$\varphi(0) = \sigma(B) = (\lambda_1, 0, 0), \text{ and } \varphi_3'(0) = 0.$$

Assume we have a map $\varphi \in \mathcal{O}(\mathbb{D}, \mathbb{G}_3)$ such that (4.3) holds. Set

$$\psi(\zeta) := \begin{pmatrix} 0 & \zeta & 0 \\ 0 & 0 & \zeta \\ \zeta^{-2}\varphi_3(\zeta) & -\zeta^{-1}\varphi_2(\zeta) & \varphi_1(\zeta) \end{pmatrix}.$$

Then $\sigma \circ \psi = \varphi$, $\psi(0)$ is conjugate to B (not necessarily equal to it), and for any $\zeta \neq 0$, e_3 is a cyclic vector for $\psi(\zeta)$.

5. Examples

Recall from [11, Proposition 4] that when $A = tI$, the function $l_{\Omega_n}(., B)$ is continuous at A, or equivalently the function l_{Ω_n} is continuous at (A, B), if and only if all the eigenvalues of B are equal. For $n = 2$, this covers all the derogatory cases. Using an automorphism of Ω_n, the situation quickly reduces to the case $t = 0$. The next example in the case $n = 3$ is then

$$A := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Example 5.1. *Taking*

$$B = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & j\varepsilon & 0 \\ 0 & 0 & j^2\varepsilon \end{pmatrix}, \ where \ \varepsilon > 0, j = -1/2 + i\sqrt{3}/2$$

and ε small enough, the function $l_{\Omega_n}(.,B)$ is discontinuous at A.

Indeed, we clearly have $spA = \{0\}, \sigma(A) = (0,0,0)$; B is non-derogatory. The eigenvalues of B are ε; $j\varepsilon$ and $j^2\varepsilon$. Thus $\sigma(B) = (0, 0, \varepsilon^3)$.

We can find $r > 0$ such that $\mathbb{B}(0,r) \subset \mathbb{G}_3$, where $\mathbb{B}(0,r)$ denotes the Euclidean ball with center at 0 and radius r. For ε chosen small enough, $\sigma(B) \in \mathbb{B}(0,r)$. By the definition of Lempert function and [8, Proposition 3.1.10], we conclude that

$$l_{\mathbb{G}_3}(\sigma(A), \sigma(B)) = l_{\mathbb{G}_3}(0, \sigma(B)) \leq l_{\mathbb{B}(0,r)}(0, \sigma(B)) = \frac{\|\sigma(B)\|}{r} = \frac{\varepsilon^3}{r}.$$

On the other hand, if there is an analytic function $\varphi : \mathbb{D} \to \Omega_3$ such that $\varphi(0) = A$ and $\varphi(\zeta) = B$ then, by (3.1) we have $\varepsilon^2 = \max\{\varepsilon^2, \varepsilon^3\} \leq |\zeta|$. It follows that

$$l_{\Omega_3}(A, B) \geq \varepsilon^2 > \frac{\varepsilon^3}{r} \geq l_{\mathbb{G}_3}(\sigma(A), \sigma(B))$$

for ε is small enough.

Example 5.2. *If the eigenvalues of B are equal, then the function $l_{\Omega_n}(.,B)$ is continuous at A (moreover l_{Ω_n} is continuous at (A, B)).*

Indeed, if the eigenvalues of B are equal, say to μ, then, by [11] and (1.1) we have

$$|\mu| \geq l_{\Omega_3}(A, B) \geq l_{\mathbb{G}_3}(\sigma(A), \sigma(B)).$$

On the other hand, if $C_{\mathbb{G}_3}$ is the Carathéodory pseudodistance of $\mathbb{G}_3$ then

$$l_{\mathbb{G}_3}((0,0,0), (3\mu, 3\mu^2, \mu^3)) = l_{\mathbb{G}_3}(\sigma(A), \sigma(B)) \geq C_{\mathbb{G}_3}(\sigma(A), \sigma(B))$$

$$\geq \sup_{|\lambda|=1} \left| \frac{f_\lambda(\sigma(B)) - f_\lambda(\sigma(A))}{1 - \overline{f_\lambda(\sigma(A))}f_\lambda(\sigma(B))} \right| = |\mu|,$$

where $f_\lambda(S) = \dfrac{s_1 + 2s_2\lambda + 3s_3\lambda^2}{3 + 2s_1\lambda + s_2\lambda^2}, \forall S = (s_1, s_2, s_3) \in \mathbb{G}_3, \lambda \in \overline{\mathbb{D}}$, (for the last inequality see [4] or [9]). Thus $l_{\Omega_3}(A, B) = l_{\mathbb{G}_3}(\sigma(A), \sigma(B)) = |\mu|$.

6. The Pluricomplex Green and Lempert functions are not equal

Let D be a domain in $\mathbb{C}^n$. We call Green function the exponential of the usual Green function with logarithmic singularity.

Definition 6.1. *For $(a, z) \in D \times D$, the pluricomplex Green function with pole at a, evaluated at z is*

$$g_D(a, z) := \sup\{u(z) : u : D \to [0, 1), \log u \in PSH(D),$$
$$\exists C = C(u, a) > 0, \forall w \in D : u(w) \leq C\|w - a\|\},$$

where $PSH(D)$ denotes the family of all functions plurisubharmonic on D (and $\|\ \|$ is the Euclidean norm in $\mathbb{C}^n$).

The formulas for the Carathéodory and the Lempert functions on $\mathbb{G}_2$ were obtained by Agler and Young [2]. Using the fact that $C_{\mathbb{G}_2} = l_{\mathbb{G}_2}$, Costara [5] has obtained a formula for the Carathéodory and the Lempert functions on Ω_2. He proved that on Ω_2 the Carathéodory and the Lempert functions do not coincide. The same holds for the Green function, for any n.

Proposition 6.2. *Let A be a cyclic matrix in Ω_n such that at least two of the eigenvalues of A are not equal. Then there exists a matrix $B \in \Omega_n$ such that*

$$l_{\Omega_n}(A, B) > g_{\Omega_n}(A, B).$$

For Green functions with several poles, there are cases of strict inequality even for the usual bidisk [12].

Proof. We need the following.

Proposition 6.3 (Edigarian-Zwonek). *Let $A, B \in \Omega_n$. Then*

$$(6.1) \qquad g_{\Omega_n}(A, B) = g_{\Omega_n}(A, diag(\mu_1, \mu_2, \cdots, \mu_n)),$$

where $sp(B) = \{\mu_j, 1 \leq j \leq n\}$, with the eigenvalues repeated according to multiplicity.

Take $B_\alpha = (b_{i,j})_{1 \leq i,j \leq n} \in \Omega_n$, where $b_{1,1} = b_{2,2} = \cdots = b_{n,n} = \mu \in \mathbb{D}; b_{j-1,j} = \alpha \in \mathbb{C}, \forall 2 \leq j \leq n$, and all other coefficients $b_{i,j} = 0, 1 \leq i, j \leq n$.

Since B_0 is a scalar matrix, by using [10], we have discontinuity of the Lempert function l_{Ω_n} at (A, B_0). Then by Proposition 1.2

$$l_{\Omega_n}(A, B_0) > l_{G_n}(\sigma(A), \sigma(B_0)).$$

Consider now $\alpha \neq 0$. Then A, B_α are cyclic matrices. It implies that

$$l_{G_n}(\sigma(A), \sigma(B_0)) = l_{G_n}(\sigma(A), \sigma(B_\alpha)) = l_{\Omega_n}(A, B_\alpha).$$

On the other hand

$$l_{\Omega_n}(A, B_\alpha) \geq g_{\Omega_n}(A, B_\alpha) = g_{\Omega_n}(A, B_0),$$

where the last equality follows from (6.1).

Thus $l_{\Omega_n}(A, B_0) > g_{\Omega_n}(A, B_0)$.

$\square$

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Université de Toulouse, UPS, INSA, UT1, UTM, Institut de Mathématiques de Toulouse, F-31062 Toulouse, France

E-mail address: `pthomas@math.univ-toulouse.fr`

Department of Mathematics, Hanoi National University of Education, 136 Xuan Thuy str - Cau Giay, Hanoi - Vietnam

E-mail address: `ngvtrao@yahoo.com`